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Magnetic charges in the AdS_4 superalgebra $\text{osp}(4|2)$

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ABSTRACT: We discuss the issue of how to include magnetic charges in the AdS_4 superalgebra $\text{osp}(4|2)$. It is shown that the usual way of introducing a pseudoscalar central charge on the right hand side of the basic anticommutator does not work, because this breaks $\text{SO}(2,3)$ covariance. We propose a way out by promoting the magnetic charge to a vector charge, which amounts to enlarge $\text{osp}(4|2)$ to the superconformal algebra $\text{su}(2,2|1)$. The conditions for $1/4$, $1/2$ and $3/4$ BPS states are then analyzed. These states form the boundary of the convex cone associated with the Jordan algebra of 4×4 complex hermitian matrices. An Inönü-Wigner contraction of the constructed superalgebra yields a known extension of the Poincaré superalgebra containing electric and magnetic 0-brane charges as well as string- and space-filling 3-brane charges. As an example, we show how some supersymmetric AdS_4 black holes fit into the classification scheme of BPS states.

KEYWORDS: Extended Supersymmetry, AdS-CFT Correspondence, Black Holes

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1 Introduction

It is well-known that in asymptotically AdS₄ spacetimes there are extremal supersymmetric black holes carrying magnetic charges [1, 2].¹ In order to describe such objects we have to find an extension of $\text{osp}(4|2)$ that includes additional generators corresponding to such charges. A first attempt to do this can be found in [4], where an extra generator V (representing the magnetic charge) was added to $\text{osp}(4|2)$ by deforming the basic anticommutator in the following way:

$$\{Q_\alpha^i, Q_\beta^j\} = \delta^{ij} \left((\gamma^a M_{a4} - \gamma^{ab} M_{ab}) C \right)_{\alpha\beta} + i\epsilon^{ij} (C_{\alpha\beta} U + i(C\gamma^5)_{\alpha\beta} V) , \quad (1.1)$$

where U denotes the electric charge. The problem with this is that one of the super-Jacobi identities fails to hold. Indeed one finds²

$$\{Q_\alpha^i, [Q_\beta^j, M_{AB}]\} + \text{permutations} = \frac{1}{2} V \epsilon^{ij} ((\Gamma_{AB} \Gamma^4 C^{-1})_{\beta\alpha} - (\Gamma_{AB} \Gamma^4 C^{-1})_{\alpha\beta}) ,$$

¹The earliest reference in this topic is [3], but the BPS magnetic monopoles found there have naked singularities, and are thus not really black holes.

²Capital latin indices A, B, \dots refer to $\text{SO}(2, 3)$ tensors and Γ_A are $\text{SO}(2, 3)$ Dirac matrices, cf. appendix A. M_{AB} denote the $\text{SO}(2, 3)$ generators, that are split in (1.1) into M_{ab} and M_{a4} , with $a = 0, \dots, 3$.

which is clearly nonvanishing. This should not be surprising, since the magnetic term $\gamma^5 V$ in (1.1) breaks $\text{SO}(2,3)$ covariance. (γ^5 is essentially one of the $\text{SO}(2,3)$ generators). In section 2 we will propose a way out by promoting the magnetic charge to a vector charge, and show that this construction amounts to enlarging $\text{osp}(4|2)$ to the superconformal algebra $\text{su}(2,2|1)$. In 3 we shall analyze the conditions for BPS states of this algebra, and compare the results with some known supersymmetric black hole solutions. In the following section, the geometrical interpretation of these BPS states is discussed, and it is shown that they form the boundary of the convex cone associated with the Jordan algebra $J_4^{\mathbb{C}}$ of 4×4 complex hermitian matrices. We conclude in 5 with some final remarks. Appendix A contains our notations and conventions, while in appendix B we briefly review the superalgebra $\text{osp}(4|N)$.

Here we shall not try to include nut charges. A way to do this in the case of the $N = 2$ Poincaré superalgebra in four dimensions was proposed in [5].³ Since there exist also several supersymmetric black hole solutions in AdS with nonvanishing nut charges [7], it would be very interesting to see how AdS superalgebras take them into account. We hope to come back to this point in a future publication.

2 Construction of the superalgebra

In order to restore $\text{SO}(2,3)$ covariance, let us modify the basic anticommutator of $\text{osp}(4|2)$ according to

$$\{Q_\alpha^i, Q_\beta^j\} = \frac{1}{2} \delta^{ij} (\Gamma^{AB} C^{-1})_{\alpha\beta} M_{AB} + (C^{-1})_{\alpha\beta} U \epsilon^{ij} + \epsilon^{ij} (\Gamma^A C^{-1})_{\alpha\beta} V_A, \quad (2.1)$$

where Q_α^i ($i = 1, 2$) are Majorana spinors and V_A represent now magnetic vector charges. Using the representation of the $\text{SO}(2,3)$ gamma matrices Γ^A in terms of $\text{SO}(1,3)$ Dirac matrices γ^a given in appendix A, we get

$$\Gamma^A C^{-1} V_A = \gamma^5 C^{-1} V_4 + \gamma^5 \gamma^a C^{-1} V_a,$$

and hence the first term on the rhs corresponds to the extension (1.1) proposed in [4], whereas the second one is necessary for $\text{SO}(2,3)$ covariance. Provided that $V^2 \equiv V^A V_A$ is negative, one can always set $V_a = 0$ by an $\text{SO}(2,3)$ transformation, so that in this case (1.1) is just a sort of gauge-fixed version of (2.1). Of course, such a gauge fixing spoils covariance under $\text{SO}(2,3)$, which is broken down to the Lorentz group $\text{SO}(1,3)$.

The remaining commutation relations can be fixed by imposing the super-Jacobi identities. This leads to

$$\begin{aligned} [M_{AB}, M_{CD}] &= \eta_{AC} M_{BD} + \eta_{BD} M_{AC} - \eta_{AD} M_{BC} - \eta_{BC} M_{AD}, \\ [M_{AB}, Q_\alpha^i] &= \frac{1}{2} (\Gamma_{AB})_\alpha{}^\beta Q_\beta^i, & [U, Q_\alpha^k] &= -\frac{3}{2} \epsilon_{ij} \delta^{jk} Q_\alpha^i, \\ [V_A, Q_\alpha^i] &= \frac{1}{2} (\Gamma_A)_\alpha{}^\beta \delta^{ik} \epsilon_{kj} Q_\beta^j, & [V_A, V_B] &= M_{AB}, \\ [M_{AB}, V_C] &= \eta_{AC} V_B - \eta_{BC} V_A, & [U, M_{AB}] &= [U, V_A] = 0. \end{aligned} \quad (2.2)$$

³Cf. also [6].

This superalgebra is actually isomorphic to the superconformal algebra $\text{su}(2, 2|1)$ in four dimensions. The latter, which has the generators $P_a, J_{ab}, D, K_a, Q_\alpha, S_\alpha$ plus an internal $U(1)$ symmetry generator A , is given by the Lorentz group together with [8]

$$\begin{aligned}
 [J_{ab}, P_c] &= \eta_{bc}P_a - \eta_{ac}P_b, & [J_{ab}, K_c] &= \eta_{bc}K_a - \eta_{ac}K_b, \\
 [D, P_a] &= -P_a, & [D, K_a] &= K_a, \\
 [P_a, K_b] &= -2J_{ab} + 2\eta_{ab}D, & [K, K] &= [P, P] = 0, \\
 [Q_\alpha, J_{ab}] &= \frac{1}{2}(\gamma_{ab})_\alpha{}^\beta Q_\beta, & [S_\alpha, J_{ab}] &= \frac{1}{2}(\gamma_{ab})_\alpha{}^\beta S_\beta, \\
 \{Q_\alpha, Q_\beta\} &= -2(\gamma^a C^{-1})_{\alpha\beta} P_a, & \{S_\alpha, S_\beta\} &= 2(\gamma^a C^{-1})_{\alpha\beta} K_a, \\
 [Q_\alpha, D] &= \frac{1}{2}Q_\alpha, & [S_\alpha, D] &= -\frac{1}{2}S_\alpha, \\
 [Q_\alpha, K_a] &= -(\gamma_a)_\alpha{}^\beta S_\beta, & [S_\alpha, P_a] &= (\gamma_a)_\alpha{}^\beta Q_\beta, \\
 [Q_\alpha, A] &= -\frac{3i}{4}(\gamma^5)_\alpha{}^\beta Q_\beta, & [S_\alpha, A] &= \frac{3i}{4}(\gamma^5)_\alpha{}^\beta S_\beta, \\
 \{Q_\alpha, S_\beta\} &= -2(C^{-1})_{\alpha\beta} D + (\gamma^{ab} C^{-1})_{\alpha\beta} J_{ab} + 4i(\gamma^5 C^{-1})_{\alpha\beta} A.
 \end{aligned} \tag{2.3}$$

The isomorphism with (2.1), (2.2) is provided by

$$\begin{aligned}
 M_{ab} &= -J_{ab}, & M_{a4} &= -\frac{1}{2}(P_a - K_a), & V_a &= -\frac{1}{2}(P_a + K_a), & V_4 &= D, \\
 Q^1 &= \frac{1}{2}(Q - S), & Q^2 &= -\frac{1}{2}\gamma^5(Q + S), & U &= 2iA.
 \end{aligned} \tag{2.4}$$

It is interesting to perform an Inönü-Wigner contraction of the superalgebra (2.1), (2.2). This is done by rescaling

$$Q_\alpha^i \rightarrow \lambda^{1/2} Q_\alpha^i, \quad M_{a4} \rightarrow \lambda M_{a4}, \quad U \rightarrow \lambda U, \quad V_A \rightarrow \lambda V_A, \tag{2.5}$$

and then taking the limit $\lambda \rightarrow \infty$, yielding

$$\begin{aligned}
 \{Q_\alpha^i, Q_\beta^j\} &= -\delta^{ij}(\gamma^a C^{-1})_{\alpha\beta} P_a + \epsilon^{ij}(\gamma^5 \gamma^a C^{-1})_{\alpha\beta} V_a + \epsilon^{ij}U(C^{-1})_{\alpha\beta} + \epsilon^{ij}V_4(\gamma^5 C^{-1})_{\alpha\beta}, \\
 [M_{ab}, M_{cd}] &= \eta_{ac}M_{bd} + \eta_{bd}M_{ac} - \eta_{ad}M_{bc} - \eta_{bc}M_{ad}, \\
 [M_{ab}, P_c] &= \eta_{ac}P_b - \eta_{bc}P_a, & [M_{ab}, V_c] &= \eta_{ac}V_b - \eta_{bc}V_a, \\
 [M_{ab}, Q_\alpha^i] &= \frac{1}{2}(\gamma_{ab})_\alpha{}^\beta Q_\beta^i
 \end{aligned} \tag{2.6}$$

as the only nonvanishing (anti)commutation relations.⁴ (2.6) is a subcase of the extended Poincaré superalgebra considered in [9], with the spatial components of V_a representing string charges, whereas V_0 is a charge for space-filling 3-branes [6]. In the AdS version (2.1), (2.2), these string- and brane charges are thus unified with the magnetic 0-brane charge in the vector V_A . Note in this context that the inclusion of brane charges in AdS_5 superalgebras was discussed in [10, 11].

⁴Here we defined $P_a = -M_{a4}$.

3 Analysis of BPS states

In our conventions, Majorana spinors are real. Hence, we can define a single complex Dirac supercharge by

$$Q = \frac{1}{\sqrt{2}}(Q^1 + iQ^2), \quad (3.1)$$

and the only nontrivial anticommutator (cf. (2.1)) becomes

$$\begin{aligned} \{Q_\alpha, Q_\beta^*\} = & \frac{1}{2} \left(\gamma^{ab} C^{-1} \right)_{\alpha\beta} M_{ab} - \left(\gamma^a C^{-1} \right)_{\alpha\beta} P_a - i \left(C^{-1} \right)_{\alpha\beta} U \\ & - i \left(\gamma^5 \gamma^a C^{-1} \right)_{\alpha\beta} V_a - i \left(\gamma^5 C^{-1} \right)_{\alpha\beta} V_4, \end{aligned} \quad (3.2)$$

where we have set $M_{a4} = -P_a$. When there is a multiplet of BPS states, some combinations of the supercharges have to be represented trivially, i.e., they have to vanish. This means that the positive semi-definite hermitian matrix $\{Q_\alpha, Q_\beta^*\}$ is not of maximal rank; the right hand side of (3.2) must have at least one vanishing eigenvalue.

In order to compute the determinant of $\{Q, Q^*\}$, we choose the explicit representation for the gamma matrices given in appendix A, and define

$$J^i = -\frac{1}{2} \epsilon^{ijk} M_{jk}, \quad K^i = -M^{0i}, \quad H = P^0, \quad V^A = (V^0, \mathbf{V}, W). \quad (3.3)$$

Note that $\det\{Q, Q^*\}$ is manifestly $\text{SL}(4, \mathbb{C})$ invariant, but the subgroup keeping H fixed is its maximal compact $\text{SU}(4)$ subgroup. $\det\{Q, Q^*\}$ is thus a fourth-order polynomial in H with coefficients that are homogeneous polynomials in the three algebraically-independent $\text{SU}(4)$ invariants that can be constructed from $\mathbf{J}, \mathbf{P}, \mathbf{K}, \mathbf{V}, U, V^0$ and W . In fact we find⁵

$$\det(\{Q, Q^*\}) = H^4 - 2aH^2 + 8bH + (a^2 - 4c), \quad (3.4)$$

where

$$a = |\mathbf{J}|^2 + |\mathbf{P}|^2 + |\mathbf{K}|^2 + |\mathbf{V}|^2 + U^2 + (V^0)^2 + W^2, \quad (3.5)$$

$$b = \mathbf{P} \cdot (\mathbf{J} \times \mathbf{K}) + \mathbf{V} \cdot (U\mathbf{J} + W\mathbf{K} + V^0\mathbf{P}), \quad (3.6)$$

$$\begin{aligned} c = & ((V^0)^2 + W^2 + U^2) |\mathbf{V}|^2 + |U\mathbf{J} + W\mathbf{K} + V^0\mathbf{P}|^2 + \\ & + |\mathbf{V} \times \mathbf{P}|^2 - 2\mathbf{V} \cdot (U\mathbf{P} \times \mathbf{K} + V^0\mathbf{K} \times \mathbf{J} + W\mathbf{P} \times \mathbf{J}) + \\ & + |\mathbf{K} \times \mathbf{P}|^2 + |\mathbf{J} \times \mathbf{P}|^2 + |\mathbf{K} \times \mathbf{J}|^2 + |\mathbf{V} \times \mathbf{J}|^2 + |\mathbf{V} \times \mathbf{K}|^2. \end{aligned} \quad (3.7)$$

The positivity condition for the hermitian matrix $\{Q, Q^*\}$ imposes a lower bound on H in terms of the invariants a, b, c ,

$$H \geq E(a, b, c), \quad (3.8)$$

where E is the largest root of (3.4), which has to be non-negative since the sum of the roots vanishes. (3.8) is easily seen by writing (3.2) in the form

$$\{Q_\alpha, Q_\beta^*\} = H\delta_{\alpha\beta} - \Lambda_{\alpha\beta},$$

⁵The absence of the H^3 term in (3.4) is due to the fact that it represents the characteristic polynomial of a traceless matrix.

and going to a basis in which $\Lambda_{\alpha\beta}$ is diagonal. Configurations saturating (3.8) are BPS, the number of supersymmetries preserved being equal to the multiplicity of the eigenvalue E . Since the characteristic polynomial (3.4) is the same as the one for the centrally extended $N = 1$ super-Poincaré algebra [12] (although the invariants a, b, c are of course different), and the conditions obeyed by a, b, c in order to have 1/4, 1/2 or 3/4 supersymmetry were determined in [12], we shall not repeat such an analysis here, but instead give explicit examples of black holes saturating the bound (3.8).

3.1 Supersymmetric AdS black holes in $N = 2$ gauged supergravity

We shall now compare the above results with some known BPS black hole solutions in minimal $N = 2$ gauged supergravity [1, 4]. We would like to stress that the present classification only analyzes the supersymmetry algebra and thus it provides model-independent constraints on BPS configurations but it is still regardless of possible further constraints imposed by certain realizations thereof.

3.1.1 The Kerr-Newman-AdS₄ solution

The metric and electromagnetic vector potential of the Kerr-Newman-AdS₄ black hole are given respectively by (cf. e.g. [13])

$$\begin{aligned} ds^2 &= -\frac{\Delta_r}{\rho^2} \left(dt - \frac{a \sin^2 \theta}{\Xi} d\varphi \right)^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left(a dt - \frac{r^2 + a^2}{\Xi} d\varphi \right)^2, \\ A &= -\frac{q_e r}{\rho^2} \left(dt - \frac{a \sin^2 \theta}{\Xi} d\varphi \right) - \frac{q_m \cos \theta}{\rho^2} \left(a dt - \frac{r^2 + a^2}{\Xi} d\varphi \right), \end{aligned} \quad (3.9)$$

where

$$\begin{aligned} \rho^2 &= r^2 + a^2 \cos^2 \theta, & \Delta_\theta &= 1 - \frac{a^2}{l^2} \cos^2 \theta, \\ \Delta_r &= (r^2 + a^2) \left(1 + \frac{r^2}{l^2} \right) - 2mr + q_e^2 + q_m^2, & \Xi &= 1 - \frac{a^2}{l^2}. \end{aligned}$$

The parameters m, a, q_e and q_m are related to the mass, angular momentum, electric and magnetic charges respectively⁶ (see below). This solution admits Killing spinors if [1]

$$q_m = 0, \quad m^2 = \left(1 + \frac{a}{l} \right)^2 q_e^2. \quad (3.10)$$

Plugging this into the extremality condition

$$\begin{aligned} m_{\text{extr.}} &= \frac{l}{3\sqrt{6}} \left(\sqrt{\left(1 + \frac{a^2}{l^2} \right)^2 + \frac{12}{l^2} (a^2 + q_e^2 + q_m^2) + \frac{2a^2}{l^2} + 2} \right. \\ &\quad \left. \cdot \left(\sqrt{\left(1 + \frac{a^2}{l^2} \right)^2 + \frac{12}{l^2} (a^2 + q_e^2 + q_m^2) - \frac{a^2}{l^2} - 1} \right)^{1/2} \right) \end{aligned}$$

⁶We apologize for using the same symbol a for the angular momentum parameter and the invariant (3.5), but the meaning should be clear from the context.

yields

$$m^2 = al \left(1 + \frac{a}{l}\right)^4. \quad (3.11)$$

The mass, angular momentum and electric charge of the Kerr-Newman-AdS₄ black hole are given by [13]

$$M = \frac{m}{\Xi^2}, \quad J = \frac{am}{\Xi^2}, \quad Q_e = \frac{q_e}{\Xi}. \quad (3.12)$$

Using (3.10) and (3.11) gives

$$M = \frac{\sqrt{al}}{\left(1 - \frac{a}{l}\right)^2}, \quad J = \frac{a\sqrt{al}}{\left(1 - \frac{a}{l}\right)^2}, \quad Q_e = \frac{\sqrt{al}}{\left(1 - \frac{a}{l}\right)}, \quad (3.13)$$

and thus

$$M = Q_e + \frac{J}{l}. \quad (3.14)$$

To check the consistency of this relation with the constraints coming from the superalgebra, we have to go back to (3.4) where everything is vanishing except $H = M$, $|\mathbf{J}| = J$ and $U = Q_e$. In this case, (3.4) boils down to

$$\det\{Q, Q^*\} = M^4 - 2(J^2 + Q_e^2)M^2 + (J^2 - Q_e^2)^2, \quad (3.15)$$

whose largest root is $M = Q_e + J$, which exactly coincides with (3.14) after setting $l = 1$ for the AdS curvature radius (which is the choice made in the superalgebra (2.1), (2.2)). This solution preserves one quarter of the supersymmetry.

3.1.2 Static magnetic topological black holes

Let us now consider magnetically charged BPS black holes in minimal $N = 2$, $D = 4$ gauged supergravity.⁷ These belong to a class of solutions known as topological black holes [14], whose horizon can be an arbitrary Riemann surface of constant curvature κ . In what follows we shall be interested in the case $\kappa < 0$. Then the metric and the electromagnetic gauge potential read respectively (cf. e.g. [1])

$$\begin{aligned} ds^2 &= -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2(d\theta^2 + \sinh^2\theta d\varphi^2), \\ A &= -\frac{q_e}{r}dt + q_m \cosh\theta d\varphi, \end{aligned} \quad (3.16)$$

where

$$V(r) = -1 - \frac{2m}{r} + \frac{q_e^2 + q_m^2}{r^2} + \frac{r^2}{l^2}. \quad (3.17)$$

If the horizon is compactified to a Riemann surface of genus $g > 1$, the black hole mass, electric and magnetic charges are given by

$$M = m(g-1), \quad Q_e = q_e(g-1), \quad Q_m = q_m(g-1). \quad (3.18)$$

⁷Generalizations to the case of gauged supergravity with matter coupling can be found in [2].

The Killing spinor equations for the above geometry were solved in [1], and it was found that they imply the constraints

$$m = q_e = 0, \quad q_m = \pm \frac{l}{2} \quad (3.19)$$

on the parameters. If (3.19) is satisfied, one has a 1/4 BPS, extremal massless black hole carrying only magnetic charge, with an event horizon at $r = l/\sqrt{2}$. This is a solitonic object, since it does not admit a limit where the cosmological constant $\Lambda = -3/l^2$ goes to zero. Notice that the mass appearing in (3.18) is not to be identified with the Hamiltonian $H = M_{04}$. To see this, consider the asymptotic form of (3.16) for $r \rightarrow \infty$,

$$ds^2 = - \left(-1 + \frac{r^2}{l^2} \right) dt^2 + \frac{dr^2}{-1 + \frac{r^2}{l^2}} + r^2 (d\theta^2 + \sinh^2 \theta d\varphi^2), \quad (3.20)$$

which is obtained as induced metric on the hypersurface $\eta_{AB} X^A X^B = -l^2$ in \mathbb{R}_2^5 by choosing the parametrization

$$\begin{aligned} X^0 &= r \cosh \theta, & X^1 &= r \sinh \theta \cos \varphi, & X^2 &= r \sinh \theta \sin \varphi, \\ X^3 &= \sqrt{r^2 - l^2} \cosh \frac{t}{l}, & X^4 &= \sqrt{r^2 - l^2} \sinh \frac{t}{l}. \end{aligned} \quad (3.21)$$

On the other hand, setting

$$\begin{aligned} X^0 &= \sqrt{\rho^2 + l^2} \sin \frac{\tau}{l}, & X^4 &= \sqrt{\rho^2 + l^2} \cos \frac{\tau}{l}, \\ X^1 &= \rho \sin \vartheta \cos \phi, & X^2 &= \rho \sin \vartheta \sin \phi, & X^3 &= \rho \cos \vartheta, \end{aligned} \quad (3.22)$$

one gets AdS_4 in global coordinates,

$$ds^2 = - \left(1 + \frac{\rho^2}{l^2} \right) d\tau^2 + \frac{d\rho^2}{1 + \frac{\rho^2}{l^2}} + \rho^2 (d\vartheta^2 + \sin^2 \vartheta d\phi^2). \quad (3.23)$$

The AdS Killing vectors are

$$\xi_{AB} = X_A \partial_B - X_B \partial_A.$$

In the parametrization (3.21) we have $\xi_{34} = l \partial_t$, and hence the mass (3.18), which was computed with respect to ∂_t , corresponds to the generator $M_{34} = -P_3$. The Hamiltonian $H = M_{04}$, instead, generates translations in the global time τ , since $\xi_{04} = l \partial_\tau$. In the coordinates (3.21), one has

$$\xi_{04} = \sqrt{r^2 - l^2} \sinh \frac{t}{l} \left(-\frac{\sinh \theta}{r} \partial_\theta + \cosh \theta \partial_r \right) - \frac{r \cosh \theta}{\sqrt{\frac{r^2}{l^2} - 1}} \cosh \frac{t}{l} \partial_t. \quad (3.24)$$

Setting everything to zero in (3.4) except the magnetic charge W and possible boost charges \mathbf{K} , we find that the largest root is given by

$$H = |\mathbf{K}| + |W|. \quad (3.25)$$

This is of course strictly positive, so there should be a nonvanishing conserved charge associated to (3.24). Since the holographic stress tensor of the solution (3.16) vanishes for $m = 0$ [15],⁸ it is difficult to see how such a nonzero value of H could arise. One possibility is that there exist finite counterterms that can be added to the usual counterterms used in the holographic renormalization procedure (cf. e.g. [16]), which would then contribute to the stress tensor and thus to H .

Apart from this issue, it is also clear from (3.25) that we need $\mathbf{K} \neq 0$ (alternatively one could have $\mathbf{P} \neq 0$), since otherwise (3.25) becomes a double root, but we know from the Killing spinor equations that the above black hole preserves only one quarter of the supersymmetry [1]. This seems to suggest that the solution (3.16) indeed carries boost charges as well. This is not incongruous, since in the description in terms of Poincaré coordinates, t in (3.16) becomes Rindler time [15], and a translation in Rindler time corresponds to a boost in Minkowski time. We shall leave a deeper investigation of these points for a future publication.

Notice finally that (3.19) gives a Dirac-type quantization condition for the magnetic charge, which comes from the minimal coupling of the gravitino to the gauge field (with coupling constant $1/l$), and does not seem to have any model-independent origin.

4 Geometrical interpretation of BPS states

In this section we shall discuss the geometry associated with BPS representations of the algebra (2.1), (2.2). To this end, we need the concepts of convex cones and Jordan algebras, that will be introduced below. Our presentation follows closely ref. [12], to which we refer for more details.

4.1 Convex cones

Definition. Let V be a real vector space. A convex cone $\mathcal{C} \subset V$ is an n -dimensional subspace such that

$$\text{i) } x \in \mathcal{C}, \lambda \in \mathbb{R}^+ \Rightarrow \lambda x \in \mathcal{C}, \quad \text{ii) } x, y \in \mathcal{C} \Rightarrow x + y \in \mathcal{C} .$$

If $\langle, \rangle : V \times V^* \rightarrow \mathbb{R}$ is a bilinear map, one can define the dual cone \mathcal{C}^* by

$$\mathcal{C}^* := \{y \in V^* | \langle x, y \rangle > 0 \ \forall x \in \mathcal{C}\} . \quad (4.1)$$

Once V^* is equipped with a translation-invariant measure $d^n y$, it becomes possible to define a characteristic function ω associated with the convex cone in the following way:

$$\omega^{-1}(x) := \int_{\mathcal{C}^*} e^{-\langle x, y \rangle} d^n y . \quad (4.2)$$

⁸Actually, [15] considers only uncharged topological black holes. However, since the charges q_e, q_m give only subleading contributions to the metric compared with the mass term, they do not alter the holographic stress tensor.

Since the map $D : x \mapsto \lambda x$, $\lambda \in \mathbb{R}^+$, is an automorphism of \mathcal{C} , ω is a homogeneous function of degree n . The immediate proof of this statement consists in a simple change of variables in the integral (4.2). A consequence of this observation is $\langle x, \pi(x) \rangle = 1 \ \forall x \in \mathcal{C}$, where

$$\pi(x) := \frac{1}{n} \frac{\partial \log \omega}{\partial x} . \quad (4.3)$$

Thus, $\pi \in \mathcal{C}^*$, and by letting x range over all vectors in \mathcal{C} , $\pi(x)$ spans the whole \mathcal{C}^* . One can now introduce a metric on \mathcal{C} by

$$g_{ij} = -\frac{1}{n} \partial_i \partial_j \log \omega(x) . \quad (4.4)$$

It is straightforward to shew that

$$\pi_j = x^i g_{ij} . \quad (4.5)$$

A simple example of a convex cone is the forward light cone in four-dimensional Minkowski spacetime, where $V \cong \mathbb{R}^4$ and $\omega = \mathcal{N}^2$, where $\mathcal{N}(x) = -\eta_{\mu\nu} x^\mu x^\nu$. Note that, in general, \mathcal{C} is foliated by hypersurfaces of constant ω , with the limiting case $\omega = 0$ representing the boundary of the cone. If the hypersurfaces of constant ω are homogeneous spaces, the convex cone is called homogeneous.

4.2 Jordan algebras

Definition. A Jordan algebra of dimension n and degree ν is a triplet (J, \circ, \mathcal{N}) such that

- i) J is an n -dimensional real vector space,
- ii) \circ is a commutative, power associative and bilinear product,
- iii) \mathcal{N} is a norm with the property of being a homogeneous polynomial of degree ν .

There are four infinite series of simple Jordan algebras, realizable as matrices with the Jordan product being the anticommutator: The degree 2 algebras $\Sigma(n)$ and the series $J_k^{\mathbb{R}}$, $J_k^{\mathbb{C}}$ and $J_k^{\mathbb{H}}$; in addition there is also one exceptional Jordan algebra $J_3^{\mathbb{O}}$. $\Sigma(n)$ is an n -dimensional algebra spanned by $(1, \sigma_1, \dots, \sigma_{n-1})$, where σ_a are sigma matrices of an n -dimensional Minkowski spacetime. $J_k^{\mathbb{K}}$ instead are realized by hermitian $k \times k$ matrices over the field \mathbb{K} , with norm given by the determinant.

With any Jordan algebra we can associate a selfdual homogeneous convex cone $\mathcal{C}(J)$ defined (just like in the case of Lie groups and Lie algebras) by an exponential mapping,

$$\exp : J \rightarrow \mathcal{C}(J), \quad x \mapsto \exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!} , \quad (4.6)$$

where

$$x^{k+1} = x^k \circ x, \quad x^0 = 1_J .$$

In this particular case, the characteristic function is

$$\omega = \mathcal{N}^{n/\nu} , \quad (4.7)$$

and thus the boundary of the cone is made by the elements of J with vanishing norm. The cone is foliated by copies of the homogeneous space $\text{Str}(J)/\text{Aut}(J)$, where $\text{Str}(J)$ denotes the invariance group of the norm \mathcal{N} (the structure group of the algebra), and $\text{Aut}(J)$ is the automorphism group, i.e., the subgroup of $\text{Str}(J)$ leaving the identity element 1_J invariant. In addition to $\text{Aut}(J)$ and $\text{Str}(J)$, there is a third group naturally associated to a Jordan algebra, which is the Möbius group $\text{Mo}(J)$ of fractional linear transformations,

$$J \ni X \mapsto X' = (AX + B)(CX + D)^{-1}. \quad (4.8)$$

One has therefore the sequence of groups

$$\text{Aut}(J) \subset \text{Str}(J) \subset \text{Mo}(J), \quad (4.9)$$

that can be interpreted as generalized rotation, Lorentz and conformal groups respectively [17].

As an example motivating this, let us again consider the forward light cone in four-dimensional Minkowski spacetime. This may be interpreted as the selfdual homogeneous convex cone associated with the Jordan algebra $J_2^{\mathbb{C}}$ of hermitian 2×2 matrices over \mathbb{C} by means of the identification

$$X = (X^0, \mathbf{X}) \leftrightarrow X^\mu \sigma_\mu = X^0 1 + X^i \sigma^i. \quad (4.10)$$

Since in this case the norm is given by the determinant, the structure group is $\text{SL}(2, \mathbb{C})$, which acts on 2×2 matrices by conjugation. The automorphism group fixing the identity matrix is then $\text{SU}(2)$, so that the cone is foliated by copies of $\text{SL}(2, \mathbb{C})/\text{SU}(2)$. In order to obtain the Möbius group, one imposes hermiticity of X' in (4.8), which requires that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{SU}(2, 2). \quad (4.11)$$

This is the conformal group acting on the compactification of Minkowski space. The sequence (4.9) becomes here

$$\text{SU}(2) \subset \text{SL}(2, \mathbb{C}) \subset \text{SU}(2, 2). \quad (4.12)$$

4.3 The geometry of BPS states

The basic anticommutator (3.2) can be rewritten in terms of a hermitian bispinor Z as $\{Q_\alpha, Q_\beta^*\} = Z_{\alpha\beta}$. Since $\{Q_\alpha, Q_\beta^*\}$ is positive semi-definite, Z is a vector in a convex cone, whose boundary corresponds to the BPS condition $\det Z = 0$. To see this, consider the Jordan algebra $J_4^{\mathbb{C}}$ of hermitian 4×4 matrices. Its associated convex cone \mathcal{C} consists of those complex hermitian matrices that can be written as exponentials of other hermitian matrices, or, in other words, that have only nonnegative eigenvalues. One has thus $Z \in \mathcal{C}$.

The structure group of $J_4^{\mathbb{C}}$ is $\text{SL}(4, \mathbb{C})$, whereas $\text{Aut}(J_4^{\mathbb{C}}) \cong \text{SU}(4)$, and thus the cone is foliated by copies of the symmetric space $\text{SL}(4, \mathbb{C})/\text{SU}(4)$. In order to obtain the Möbius group, we require hermiticity of X' in (4.8). This gives the conditions

$$A^\dagger C = C^\dagger A, \quad B^\dagger D = D^\dagger B, \quad A^\dagger D - C^\dagger B = 1, \quad (4.13)$$

i.e., the complex 8×8 matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(4, \mathbb{C}), \quad (4.14)$$

and the sequence (4.9) is

$$\mathrm{SU}(4) \subset \mathrm{SL}(4, \mathbb{C}) \subset \mathrm{Sp}(4, \mathbb{C}). \quad (4.15)$$

The relevance of the Möbius group lies in its interpretation as invariance group of the BPS condition $\det Z = 0$ [12]: For the standard $D = 4$, $N = 1$ Poincaré superalgebra without central charges, all BPS states obey $P^2 = 0$. This is the momentum space version of the massless wave equation, which is invariant under the conformal group $\mathrm{SU}(2, 2)$ in $3 + 1$ dimensions. The identification of four-dimensional Minkowski space with the Jordan algebra of hermitian 2×2 matrices leads then to an identification of this conformal group with the Möbius group of $J_2^{\mathbb{C}}$. As a generalization of this, we have thus found that the BPS condition for the AdS superalgebra (2.1), (2.2) is preserved by the symplectic group $\mathrm{Sp}(4, \mathbb{C})$.

In what follows, we would like to give a geometrical interpretation of the various subsets of the cone containing states that preserve different fractions of supersymmetry. To this end, we first note that, as in [12], the cone is a stratified space with strata \mathcal{S}_n , $n = 0, 1, 2, 3, 4$, where \mathcal{S}_n denotes the subspace in which at least n of the four eigenvalues vanish, corresponding to at least n supersymmetries being preserved, and \mathcal{S}_{n+1} is the boundary of \mathcal{S}_n . \mathcal{S}_0 is the convex cone itself, consisting of all positive hermitian 4×4 matrices. The boundary of the cone is the subspace \mathcal{S}_1 , which is the 15-dimensional space of matrices of rank 3 or less. The boundary of this, containing the states preserving at least one half of the supersymmetries, is the space \mathcal{S}_2 of matrices of rank 2 or less, which has dimension 14. In order to see this, recall that a matrix of rank two is completely specified by its two normalized eigenvectors belonging to the nonvanishing eigenvalues, plus the non-zero eigenvalues themselves. The two eigenvectors span a two-plane in \mathbb{C}^4 , i.e., they correspond to the Stiefel manifold⁹ $V_2(\mathbb{C}^4)$. Since $V_k(\mathbb{C}^n) \approx \mathrm{U}(n)/\mathrm{U}(n-k)$, one has $\dim(V_k(\mathbb{C}^n)) = 2nk - k^2$ and thus $\dim(V_2(\mathbb{C}^4)) = 12$. Taking into account also the two eigenvalues, we get

$$\mathcal{S}_2 \approx (\mathrm{U}(4)/\mathrm{U}(2)) \oplus (\mathbb{R}^+)^2,$$

that has dimension 14, as claimed above. The boundary of \mathcal{S}_2 is the set \mathcal{S}_3 of matrices having rank one or less. These span a 7-dimensional space, since a rank one matrix is specified by the direction, up to a sign, of its eigenvector with non-zero eigenvalue together with the eigenvalue. This is a point in $\mathbb{C}P^3 \times \mathbb{R}^+$.

5 Final remarks

We conclude this work with some comments and possible future developments. First of all, it would be interesting to understand how the proposed superalgebra $\mathrm{su}(2, 2|1)$ fits into

⁹The Stiefel manifold $V_k(\mathbb{K}^n)$ is the set of all orthonormal k -frames in \mathbb{K}^n .

$N = 2$ gauged supergravity, whose supersymmetric solutions were studied as examples. Our results might in fact point towards a hidden superconformal invariance of this supergravity theory. In this context it is amusing to note that the superconformal group plays a role in the construction of $N = 2$, $D = 4$ (gauged) supergravity (cf. [18] for a review). This goes under the name of superconformal tensor calculus.

A second point is how our proposal could be generalized to the case with more vector fields and, correspondingly, more magnetic charges, as it happens e.g. in $N = 2$ gauged supergravity coupled to vector multiplets. Black holes with more than one magnetic charge do indeed exist in these theories [2]. The most obvious thing to do would be to add a five-dimensional vector for each charge, but it remains to be seen if such a proposal fits into some adequate superalgebra. A related topic is the inclusion of magnetic charges in $\text{osp}(4|N)$ for $N > 2$, which should lead to extended superconformal algebras. A further investigation of these points will be presented elsewhere.

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A Conventions

Throughout this work, indices i, j, \dots range from 1 to 2, and $\epsilon_{ij} = -\epsilon_{ji}$, with $\epsilon_{12} = 1$. Capital latin indices $A, B, \dots = 0, \dots, 4$ refer to $\text{SO}(2, 3)$ tensors. The gamma matrices Γ_{AB} satisfy

$$\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}, \quad (\text{A.1})$$

where $\eta_{AB} = \text{diag}(-1, 1, 1, 1, -1)$. For the Dirac matrices γ^a ($a = 0, \dots, 3$) in four dimensions we choose the real representation

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}, & \gamma^1 &= \begin{pmatrix} -\sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}, & \gamma^2 &= \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}, \\ \gamma^3 &= \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}, & \gamma^5 &= \gamma^{0123} = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}, \end{aligned}$$

where σ^i denote the standard Pauli matrices. Starting from this, one can construct a realization of the $\text{SO}(2, 3)$ Clifford algebra (A.1) by setting

$$\Gamma^A = \begin{cases} \gamma^5 \gamma^a, & A = a = 0, \dots, 3, \\ \gamma^5, & A = 4. \end{cases} \quad (\text{A.2})$$

This implies

$$\Gamma^{AB} \equiv \frac{1}{2}[\Gamma^A, \Gamma^B] = \begin{cases} \gamma^{ab}, & A = a, B = b, \\ \gamma^a, & A = a, B = 4. \end{cases} \quad (\text{A.3})$$

The charge conjugation matrix C in four dimensions satisfies $C^T = C^{-1} = -C$ and

$$\gamma^{aT} = -C\gamma^a C^{-1} . \quad (\text{A.4})$$

Using the definition (A.2), one shows that C is then a charge conjugation matrix in five dimensions as well, but with a change of sign,

$$\Gamma^{AT} = C\Gamma^A C^{-1} . \quad (\text{A.5})$$

Here we choose $C = \gamma^0$. Then the Majorana condition $\bar{\epsilon} = \epsilon^T C$, where $\bar{\epsilon} = \epsilon^\dagger \gamma^0$, implies that the spinor ϵ is real. Notice that the Majorana condition is preserved under $\text{SO}(2,3)$ transformations, since (A.2), together with

$$\gamma^{a*} = B\gamma^a B^{-1} , \quad (\text{A.6})$$

where $B = -C\gamma^0$, implies

$$\Gamma^{AB*} = B\Gamma^{AB} B^{-1} . \quad (\text{A.7})$$

B The superalgebra $\text{osp}(4|N)$

The superalgebra $\text{osp}(4|N)$ is defined as the set of graded $(4+N) \times (4+N)$ matrices μ satisfying the conditions

$$\begin{aligned} \mu^T \begin{pmatrix} C & 0 \\ 0 & 1_{N \times N} \end{pmatrix} + \begin{pmatrix} C & 0 \\ 0 & 1_{N \times N} \end{pmatrix} \mu &= 0 , \\ \mu^\dagger \begin{pmatrix} \gamma^0 & 0 \\ 0 & -1_{N \times N} \end{pmatrix} + \begin{pmatrix} \gamma^0 & 0 \\ 0 & -1_{N \times N} \end{pmatrix} \mu &= 0 . \end{aligned}$$

These equations can be solved by setting

$$\mu = \begin{pmatrix} \frac{1}{4}\epsilon^{AB}[\Gamma_A, \Gamma_B] & \chi^i \\ \bar{\chi}^i & i\epsilon_{ij}t^{ij} \end{pmatrix} , \quad (\text{B.1})$$

where ϵ^{AB} and ϵ_{ij} denote respectively an arbitrary real antisymmetric 4×4 and $N \times N$ tensor, Γ_A are $\text{SO}(2,3)$ Dirac matrices, t^{ij} represent $\text{SO}(N)$ generators and χ^i is a set of N Majorana spinors. The bosonic subalgebra of $\text{osp}(4|N)$ is $\text{sp}(4) \oplus \text{so}(N) \cong \text{so}(2,3) \oplus \text{so}(N)$. In the case $N = 2$ the $\text{so}(2)$ generator is interpreted as electric charge.

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